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A formula for polynomials with Hermitian matrix argument ☆

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Abstract

We construct and study orthogonal bases of generalized polynomials on the space of Hermitian matrices. They are obtained by the Gram–Schmidt orthogonalization process from the Schur polynomials. A Berezin–Karpelevich type formula is given for these multivariate polynomials. The normalization of the orthogonal polynomials of Hermitian matrix argument and expansions in such polynomials are investigated.

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1. Introduction

We present first the main object of this paper, the generalized orthogonal polynomials with Hermitian matrix argument. They are orthogonal with respect to measures constructed in the following way from measures on $\mathbb{R}$.

Let H_n be the space of Hermitian matrices and U_n the group of unitary matrices. We say that a function $f : H_n \to \mathbb{R}$ is central if $f(UXU^{-1}) = f(X)$ for all $U \in U_n$. Similarly, a Borel measure ν on H_n is called central if it is U_n-invariant: $\nu(UBU^{-1}) = \nu(B)$ for every Borel set $B \subset H_n$ and $U \in U_n$. If f is a central function, then f is determined by its restriction to the subspace of real diagonal matrices, which we denote D_n. Observe that $D_n \simeq \mathbb{R}^n$.

Let us define $\tilde{f}(x_1, \dots, x_n) = f(\mathrm{diag}(x_1, \dots, x_n))$. Then $\tilde{f}$ is a symmetric function in $x_1, \dots, x_n$ and the map $f \mapsto \tilde{f}$ is a bijection from the space of central functions on H_n to the space of symmetric functions on $\mathbb{R}^n$.

There is a natural and important way of generating central functions on H_n, starting from a function F on $\mathbb{R}$ by setting

$$f\big(\mathrm{diag}(x_1, \dots, x_n)\big) = F(x_1) \dots F(x_n)$$

and then extending f to a central function on H_n. One denotes $f = \det F$.

Let m be the Lebesgue measure on H_n, treated as a real vector space and let f be a positive Borel central function on H_n. We normalize m in such a way that the Weyl integration formula ([6, p. 13], [7, Th.VI.2.3]) reads

$$\int\limits_{H_n} f(X)\,dm(X) = \int\limits_{\mathbb{R}^n} f\big(\mathrm{diag}(x_1, \dots, x_n)\big) V^2(x_1, \dots, x_n)\,dx_1 \dots dx_n \tag{1}$$

where $V(x_1, \dots, x_n)$ is the Vandermonde determinant. Formula (1) implies that if G is a positive Borel function on $\mathbb{R}$, then

$$\int\limits_{H_n} f(X) \det G(X)\,dm(X)$$
$$= \int\limits_{\mathbb{R}^n} \tilde{f}(x_1, \dots, x_n) V^2(x_1, \dots, x_n) \prod_i G(x_i)\,dx_1 \dots dx_n,$$

hence the measure $\det G(X)\,dm(X)$ on H_n corresponds to the permutation invariant measure $V^2(\underline{x}) \prod_i G(x_i)\,d\underline{x}$ on $\mathbb{R}^n$. Extending this remark by duality, to any Borel measure μ on $\mathbb{R}$ we associate a permutation invariant measure μ_n on $\mathbb{R}^n$ and a central measure M on H_n in the following way:

$$\mu_n(d\underline{x}) = V^2(\underline{x})\mu^{\otimes n}(d\underline{x}); \tag{2}$$

$$\int\limits_{H_n} f(X)\,dM(X) = \int\limits_{\mathbb{R}^n} \tilde{f}(x_1, \dots, x_n) V^2(x_1, \dots, x_n)\,d\mu(x_1) \dots d\mu(x_n) \tag{3}$$

for any positive central function f on H_n.

For a symmetric polynomial P on $\mathbb{R}^n$, let $\hat{P}(X)$ be the central function on H_n whose restriction to the diagonal matrices is equal to $P(\underline{x})$. The functions $\hat{P}$ are called (generalized) polynomials of Hermitian matrix argument. In fact, $\hat{P}(X)$ is a symmetric polynomial in the eigenvalues of X.

In this article we construct and study orthogonal bases of generalized polynomials on H_n, with respect to measures M on H_n, defined in (3) for a given measure μ on $\mathbb{R}$. A Berezin–Karpelevich type formula is given for these multivariate polynomials in Theorem 3.1. The normalization of the orthogonal polynomials of Hermitian matrix argument and expansions in such polynomials are then investigated.

Let us recall that Berezin and Karpelevich [1] expressed the spherical functions on complex Grassmann manifolds $U(p,q)/U(p) \times U(q)$ as a quotient of a determinant containing Jacobi functions and a Vandermonde determinant. The Berezin–Karpelevich formula, studied by Takahashi [19], was first proved by Hoogenboom [11]. Similar formulas were given for hypergeometric functions of Hermitian matrix argument by Gross and Richards [8,9].

Our formula in Theorem 3.1 expresses the generalized orthogonal polynomials on H_n as a quotient of a determinant containing corresponding orthogonal polynomials on $\mathbb{R}$ and a Vandermonde determinant.

Generalized Hermite and Laguerre polynomials of matrix argument were introduced and studied by Herz [10]. In the Hermitian matrix case, they are orthogonal bases in the spaces $L^2_{U_n}(H_n, M)$, where the measure M is obtained as in (3) from the measure $\mu(dx) = \mathrm{e}^{-x^2}\,dx$ in the Hermite case and $\mu(dx) = x^\alpha \mathrm{e}^{-x}\mathbf{1}_{(0,\infty)}(x)\,dx$ in the Laguerre case. The notation $L^2_{U_n}$ stands for central functions in L^2. The space $L^2_{U_n}(H_n, M)$ is isomorphic to the space $L^2_{\mathrm{sym}}(\mathbb{R}^n, \mu_n)$ of symmetric functions in $L^2(\mathbb{R}^n, \mu_n)$.

More generally, the Laguerre polynomials on symmetric cones were defined in [7]. The generalized Laguerre polynomials are very useful in harmonic analysis on symmetric cones [3,7] and in multivariate statistics [16].

Hermite and Laguerre polynomials of matrix argument are special cases of generalized Hermite polynomials for Dunkl operators (cf. [17] and the references therein). They are also a special case of symmetric orthogonal polynomials associated to the Jack polynomials, studied by Lassalle in a series of notes [12–14]. In particular, Lassalle gave without proof the same formula as ours in Theorem 3.1, in the Hermite, Jacobi and Laguerre case, respectively. Our formula generalizes the formulas of Lassalle to the case of any orthogonal polynomial family on H_n, and applies also to the Hermite polynomials generalized in the sense of Chihara, Kravtchouk polynomials, Charlier, Meixner and Pollaczek polynomials, etc.

2. Preliminaries

In this section we introduce the needed notations and concepts; main references are [15] and [4]. Let us fix $n \in \mathbb{N}$. We will use n-element partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, i.e. non-increasing sequences of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Let the length $l(\lambda)$

of a partition λ be the number of $\lambda_i \neq 0$ and its degree $|\lambda| = \sum_i \lambda_i$. Our partitions λ have the length smaller or equal to n.

We will consider the dominance order between partitions of the same degree. Given two partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)$ such that $|\lambda| = |\kappa|$, we have $\lambda \geqslant \kappa$ if

$$\lambda_1 + \cdots + \lambda_r \geqslant \kappa_1 + \cdots + \kappa_r$$

for all $1 \leqslant r \leqslant n$. The dominance order is not total. The graded lexicographic order $\succ_{gl}$ on partitions is total and will also appear in the sequel. We say that $\lambda \succ_{gl} \kappa$ if $|\lambda| > |\kappa|$ or if $|\lambda| = |\kappa|$ and $\lambda_i > \kappa_i$ for the first i such that $\lambda_i \neq \kappa_i$.

Let S_n be the symmetric group of permutations of n elements. If $x_1, x_2, \dots, x_n$ are real variables, we will denote $m_\lambda(x_1, \dots, x_n)$ the monomial symmetric function in n variables

$$m_\lambda(x_1, \dots, x_n) = \sum_{\sigma \in S_n} x_1^{\lambda_{\sigma(1)}} \dots x_n^{\lambda_{\sigma(n)}}.$$

The family $\{m_\lambda\}_{l(\lambda) \leqslant n}$ is an algebraic basis of the vector space $\mathcal{P}_n$ of all symmetric polynomials in n variables.

Let $V(x_1, x_2, \dots, x_n)$ be the Vandermonde determinant,

$$V(x_1, x_2, \dots, x_n) = \det(x_j^{n-i})_{i,j=1,\dots,n} = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j). \tag{4}$$

For each partition $\lambda = (\lambda_1, \dots, \lambda_n)$, set

$$S_\lambda(x_1, \dots, x_n) = \frac{\det(x_j^{\lambda_i + n - i})_{i,j=1,\dots,n}}{V(x_1, x_2, \dots, x_n)}. \tag{5}$$

S_λ are called the Schur functions and are symmetric polynomials, homogeneous of degree $|\lambda|$. The family $\{S_\lambda\}_{l(\lambda) \leqslant n}$ is an algebraic basis of $\mathcal{P}_n$. Recall that (see [15, 7.2])

$$S_\lambda = m_\lambda + \sum_{\mu < \lambda, |\mu| = |\lambda|} k_{\lambda\mu} m_\mu, \quad k_{\lambda\mu} \in \mathbb{R}, \tag{6}$$

so the Schur polynomials are monic, in the sense that they have a dominating term m_λ with coefficient 1. Note that λ is dominating both in the dominance order and in the graded lexicographic order.

We end the introduction with the following well known properties of polynomials of n variables. We have not found a proof in the literature so we include it for the sake of completeness.

Proposition 2.1. (a) *If a polynomial $P(x_1, \dots, x_n)$ vanishes when $x_i = x_j$, then*

$$P(x_1, \dots, x_n) = (x_i - x_j) R(x_1, \dots, x_n),$$

where R is a polynomial.

(b) *If $P(x_1, \dots, x_n)$ is a polynomial vanishing when $x_i = x_j$, for all $i, j = 1, \dots, n$, $i \neq j$, then*

$$P(x_1, \dots, x_n) = V(x_1, \dots, x_n) R(x_1, \dots, x_n),$$

where R is a polynomial.

Proof. It is sufficient to consider $i = 1$ and $j = 2$. If $x_2, \ldots, x_n$ are fixed, the polynomial P is a polynomial of one variable x_1:

$$P(x_1, x_2, \ldots, x_n) = a_k(x_2, \ldots, x_n)x_1^k + a_{k-1}(x_2, \ldots, x_n)x_1^{k-1} + \cdots \\ + a_1(x_2, \ldots, x_n)x_1 + a_0(x_2, \ldots, x_n),$$

where $a_0, \ldots, a_k$ are polynomials in $x_2, \ldots, x_n$. By hypothesis,

$$0 = P(x_2, x_2, \ldots, x_n) = a_k(x_2, \ldots, x_n)x_2^k + a_{k-1}(x_2, \ldots, x_n)x_2^{k-1} + \cdots \\ + a_1(x_2, \ldots, x_n)x_2 + a_0(x_2, \ldots, x_n),$$

therefore

$$P(x_1, \ldots, x_n) = \left(x_1^k - x_2^k\right)a_k(x_2, \ldots, x_n) + \left(x_1^{k-1} - x_2^{k-1}\right)a_{k-1}(x_2, \ldots, x_n) + \cdots \\ + (x_1 - x_2)a_1(x_2, \ldots, x_n) = (x_1 - x_2)R(x_1, \ldots, x_n),$$

since $x_1^p - x_2^p = (x_1 - x_2)(x_1^{p-1} + x_1^{p-2}x_2 + \cdots + x_2^{p-1})$, for any $p \geqslant 1$. Part (b) follows immediately from part (a) of the proposition. □

3. Symmetric orthogonal polynomials

Throughout all this section, we will suppose that μ is a positive Borel measure on $\mathbb{R}$, such that:

(i) The polynomials are a dense subset of $L^2(\mathbb{R}, \mu)$.

(ii) The symmetric polynomials $\mathcal{P}_n$ are dense in $L^2_{\mathrm{sym}}(\mathbb{R}^n, \mu_n)$, where the measure $\mu_n = V^2(x_1, \ldots, x_n)\,\mu(dx_1)\ldots\mu(dx_n)$ was introduced in (2).

This is always the case when μ has an exponential moment, that is there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}} \mathrm{e}^{\varepsilon|x|}\,d\mu(x) < \infty$. In fact, if μ has an exponential moment then so does μ_n and the density of polynomials in $L^2(\mathbb{R}, \mu)$ and $L^2(\mathbb{R}^n, \mu_n)$ is well known, see [2] and [4] for a short proof. These references contain much more information on the problem of density of the space of polynomials in L^p spaces. Note also that by (i), μ must be finite.

Now we are going to construct a family $\{P_\lambda\}_{l(\lambda)\leqslant n}$ of symmetric orthogonal polynomials in n variables, starting from a family of orthogonal polynomials in one variable $\{p_m\}_{m\in\mathbb{N}}$, where p_m has degree m. This is the main result of this article.

Theorem 3.1. *Let μ be a finite positive Borel measure on $\mathbb{R}$ verifying the conditions* (i) *and* (ii) *and $\{p_m\}_{m\in\mathbb{N}}$ an orthogonal family of polynomials in $L^2(\mathbb{R}, \mu)$, where p_m has degree m. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and a normalizing constant $c_\lambda \neq 0$, let us define*

$$P_\lambda(x_1, \ldots, x_n) = c_\lambda \frac{\det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\ldots,n}}{V(x_1, \ldots, x_n)}. \tag{7}$$

Then P_λ are symmetric polynomials, orthogonal in the Hilbert space $L^2(\mathbb{R}^n, \mu_n)$, where $\mu_n = V^2(x_1, \ldots, x_n)\,\mu(dx_1)\ldots\mu(dx_n)$. The L^2-norm of P_λ is equal to

$$\|P_\lambda\|^2_{L^2(\mathbb{R}^n,\mu_n)} = c_\lambda^2 n! \prod_{i=1}^n \|p_{\lambda_i+n-i}\|^2_{L^2(\mathbb{R},\mu)}. \tag{8}$$

The family $\{P_\lambda\}_{l(\lambda)\leqslant n}$ is an orthogonal Hilbert basis of $L^2_{\mathrm{sym}}(\mathbb{R}^n,\mu_n)$, obtained by the Gram–Schmidt orthogonalization process, applied to the Schur polynomials family $\{S_\lambda\}_{l(\lambda)\leqslant n}$, ordered in the graded lexicographic order.

The normalizing constant c_λ depends on the way of normalization of P_λ and p_m and will be specified in Propositions 3.3 *and* 3.4.

Proof. First observe that

$$\det\bigl(p_{\lambda_i+n-i}(x_j)\bigr)_{i,j=1,\dots,n}=\begin{vmatrix} p_{\lambda_1+n-1}(x_1) & p_{\lambda_1+n-1}(x_2) & \dots & p_{\lambda_1+n-1}(x_n)\\ p_{\lambda_2+n-2}(x_1) & p_{\lambda_2+n-2}(x_2) & \dots & p_{\lambda_2+n-2}(x_n)\\ \vdots & \vdots & & \vdots\\ p_{\lambda_n}(x_1) & p_{\lambda_n}(x_2) & \dots & p_{\lambda_n}(x_n)\end{vmatrix}$$

is a polynomial in n variables, that vanishes for $x_i=x_j, i\neq j$. Hence, by Proposition 2.1, it is divisible by $V(x_1,\dots,x_n)$. Thus, for each λ, the function P_λ is a polynomial. Moreover P_λ is symmetric, since if $\sigma\in S_n$, denoting by $\underline{x}=(x_1,\dots,x_n)$ and $\sigma(\underline{x})=(x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)})$, we get

$$\begin{aligned}P_\lambda\bigl(\sigma(\underline{x})\bigr)&=\frac{\det(p_{\lambda_i+n-i}(x_{\sigma(j)}))_{i,j=1,\dots,n}}{\det(x^{n-i}_{\sigma(j)})_{i,j=1,\dots,n}}\\ &=\frac{\varepsilon(\sigma)\det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{\varepsilon(\sigma)\det(x_j^{n-i})_{i,j=1,\dots,n}}=P_\lambda(\underline{x}),\end{aligned}$$

where $\varepsilon(\sigma)$ denotes the signature of the permutation σ.

Now let us consider two partitions $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ and $\kappa=(\kappa_1,\kappa_2,\dots,\kappa_n)$. We have

$$\begin{aligned}\langle P_\lambda,P_\kappa\rangle_{L^2(\mathbb{R}^n,\mu_n)}&=\int\limits_{\mathbb{R}^n}P_\lambda(\underline{x})P_\kappa(\underline{x})\,d\mu_n(\underline{x})\\ &=c_\lambda c_\kappa\int\limits_{\mathbb{R}^n}\det\bigl(p_{\lambda_i+n-i}(x_j)\bigr)_{i,j=1,\dots,n}\\ &\quad\times\det\bigl(p_{\kappa_i+n-i}(x_j)\bigr)_{i,j=1,\dots,n}\,d\mu(x_1)\dots d\mu(x_n)\\ &=c_\lambda c_\kappa\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\varepsilon(\sigma)\varepsilon(\tau)\int\limits_{\mathbb{R}^n}\prod_{i=1}^n p_{\lambda_{\sigma(i)}+n-\sigma(i)}(x_i)\\ &\quad\times p_{\kappa_{\tau(i)}+n-\tau(i)}(x_i)\,d\mu(x_1)\dots d\mu(x_n)\\ &=c_\lambda c_\kappa\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\varepsilon(\sigma)\varepsilon(\tau)\prod_{i=1}^n\int\limits_{\mathbb{R}}p_{\lambda_{\sigma(i)}+n-\sigma(i)}(x)\\ &\quad\times p_{\kappa_{\tau(i)}+n-\tau(i)}(x)\,d\mu(x)\end{aligned}$$

and since $\{p_m\}_{m\in\mathbb{N}}$ is an orthogonal family in $L^2(\mathbb{R},\mu)$, if a term in the last double sum is non-zero, we must have

$$\lambda_{\sigma(i)}-\sigma(i)=\kappa_{\tau(i)}-\tau(i),\quad i=1,\dots,n.$$

Setting $\pi = \tau\sigma^{-1}$, this means that $\lambda_i - i = \kappa_{\pi(i)} - \pi(i)$ for all $i = 1, \dots, n$. It follows that if $j > i$ then $\pi(j) \geqslant \pi(i)$. In order to prove this, let us suppose that $j > i$ and $\pi(j) < \pi(i)$. We have $\kappa_{\pi(j)} \geqslant \kappa_{\pi(i)}$, that is $\lambda_j + \pi(j) - j \geqslant \lambda_i + \pi(i) - i$. This implies that $\lambda_j + \pi(j) > \lambda_i + \pi(i)$, which is contradictory with $\lambda_j \leqslant \lambda_i$ and $\pi(j) < \pi(i)$. Thus $\pi(i) \leqslant \pi(j)$ when $i < j$, what implies that necessarily π is the identity permutation. It follows that $\sigma = \tau$ and $\lambda = \kappa$. Therefore,

$$\langle P_\lambda, P_\kappa \rangle_{L^2(\mathbb{R}^n, \mu_n)} = d_\lambda \delta_{\lambda\kappa},$$

where

$$\begin{aligned} d_\lambda &= c_\lambda^2 \sum_{\sigma \in S_n} \prod_{i=1}^n \int_{\mathbb{R}} p^2_{\lambda_{\sigma(i)}+n-\sigma(i)}(x)\, d\mu(x) \\ &= c_\lambda^2 \sum_{\sigma \in S_n} \prod_{i=1}^n \| p_{\lambda_{\sigma(i)}+n-\sigma(i)} \|^2_{L^2(\mathbb{R},\mu)} \\ &= c_\lambda^2 n! \prod_{i=1}^n \| p_{\lambda_i+n-i} \|^2_{L^2(\mathbb{R},\mu)}. \end{aligned}$$

Thus $\{P_\lambda\colon\ l(\lambda) \leqslant n\}$ is an orthogonal family in $\mathcal{P}_n$ and formula (8) follows. Now, $p_{\lambda_i+n-i}(x_j) = a_{\lambda_i+n-i} x_j^{\lambda_i+n-i} +$ terms of lower degree, $a_{\lambda_i+n-i} \neq 0$, so

$$P_\lambda(\underline{x}) = a S_\lambda(\underline{x}) + \sum_{|\kappa| < |\lambda|} b_{\lambda\kappa} m_\kappa(\underline{x}), \tag{9}$$

where $a = c_\lambda \prod_i a_{\lambda_i+n-i} \neq 0$. By formulas (6) and (9) it follows that

$$\operatorname{Vect}\bigl(\{P_\lambda\}_{l(\lambda)\leqslant n}\bigr) = \operatorname{Vect}\bigl(\{S_\lambda\}_{l(\lambda)\leqslant n}\bigr) = \operatorname{Vect}\bigl(\{m_\lambda\}_{l(\lambda)\leqslant n}\bigr) = \mathcal{P}_n$$

so, by hypothesis (ii), the family $\{P_\lambda\}_{l(\lambda)\leqslant n}$ is linearly dense in $L^2_{\mathrm{sym}}(\mathbb{R}^n, \mu_n)$.

In fact, formula (9) implies a stronger fact

$$\operatorname{Vect}\bigl(\{P_\mu\}_{\mu \preceq_{gl} \lambda}\bigr) = \operatorname{Vect}\bigl(\{S_\mu\}_{\mu \preceq_{gl} \lambda}\bigr) \tag{10}$$

for any n-element partition λ. Formula (10) may be easily proved observing that $\operatorname{Vect}(\{P_\mu\}_{\mu \preceq_{gl} \lambda}) \subset \operatorname{Vect}(\{S_\mu\}_{\mu \preceq_{gl} \lambda})$ and that the dimensions of the two spaces are equal.

We order the Schur polynomials with respect to the graded lexicographic order. It follows that the family that one obtains by applying the Gram–Schmidt orthogonalization process to the family $\{S_\lambda\}$ is the family $\{P_\lambda\}$. □

Let us now extend Theorem 3.1 to polynomials of Hermitian matrix argument.

Corollary 3.2. *The generalized polynomials $\{\hat{P}_\lambda\}_{l(\lambda)\leqslant n}$ form an orthogonal Hilbert basis of the Hilbert space $L^2_{U_n}(H_n, M)$, with the measure M defined in* (3).

Now we will determine the value of the normalizing constant c_λ in the definition of P_λ, in relation with the normalization of the polynomials p_m and the required normalization of the polynomials P_λ. Formula (9) implies that in the Schur function decomposition, the polynomials P_λ have the leading term aS_λ, in the sense of the graded lexicographic order. Taking into account (6) we get

$$P_\lambda = am_\lambda + \sum_{\mu<\lambda,|\mu|=|\lambda|} ak_{\lambda\mu}m_\mu + \sum_{|\kappa|<|\lambda|} b_{\lambda\kappa}m_\kappa, \tag{11}$$

so, in monomial symmetric polynomial decomposition of P_λ, the leading term is am_λ, in the sense of both the graded lexicographic order and the graded dominance order. Consequently, it is natural to require that P_λ is monic, that is $a = 1$. From formulas (9) and (11) we deduce the following

Proposition 3.3. *If the polynomials p_m and P_λ are monic, then $c_\lambda = 1$.*

Another frequently considered type of normalization of orthogonal polynomials consists in requiring the constant term of the polynomials to be equal to 1. It is not always possible (for example for odd Hermite polynomials).

Proposition 3.4. *If we normalize the polynomials p_m and P_λ by requiring $p_m(0) = 1,\ m \in \mathbb{N}$, and $P_\lambda(\underline{0}) = 1,\ l(\lambda) \leqslant n$, then*

$$c_\lambda = \frac{1}{\det(c_{n-j}^{(\lambda_i+n-i)})_{1\leqslant i,j\leqslant n}},$$

where $c_k^{(m)}$ are the coefficients of the polynomials p_m in the monomial decomposition

$$p_m(x) = \sum_{k=0}^{m} c_k^{(m)} x^k.$$

Proof. It is shown in [6, p. 18] that if

$$f_i(x) = \sum_{k=0}^{\infty} c_k^{(i)} x^k, \quad i = 1,\dots,n,\ |x| < r,$$

then

$$\lim_{\underline{x}\to\underline{0}} \frac{\det(f_i(x_j))_{1\leqslant i,j\leqslant n}}{V(x_1,\dots,x_n)} = \det\bigl(c_{n-j}^{(i)}\bigr)_{1\leqslant i,j\leqslant n}.$$

In our case, taking

$$f_i(x) = p_{\lambda_i+n-i}(x) = \sum_{k=0}^{\lambda_i+n-i} c_k^{(\lambda_i+n-i)} x^k,$$

we have

$$P_\lambda(\underline{0}) = c_\lambda \lim_{\underline{x}\to\underline{0}} \frac{\det(f_i(x_j))_{1\leqslant i,j\leqslant n}}{V(x_1,\dots,x_n)} = c_\lambda \det\bigl(c_{n-j}^{(\lambda_i+n-i)}\bigr)_{1\leqslant i,j\leqslant n}. \qquad \square \tag{12}$$

Remark 3.5. Observe that the property $p_m(0) \neq 0$ for all $m \in \mathbb{N}$ does not imply $P_\lambda(\underline{0}) \neq 0$. Indeed, by formula (12), the value of $P_\lambda(\underline{0})$ depends not only on the coefficients $c_0 = p_m(0)$ appearing in the last column of the determinant in (12). A converse implication is however true: if λ is such that $p_{\lambda_i+n-i}(0) = 0$, $i = 1, \dots, n$, then $P_\lambda(\underline{0}) = 0$.

We end this section with a result on the expansion of some important central functions on H_n in the basis $\{P_\lambda\}$.

Proposition 3.6. *Given n functions $f_1, \dots, f_n$ of one variable, with the expansions in the basis $\{p_m\}_{m\in\mathbb{N}}$*

$$f_i(x) = \sum_{k=0}^{\infty} c_k^{(i)} p_k(x), \quad i = 1, \dots, n,$$

convergent absolutely for $|x| < r$, we have

$$\det\big(f_i(x_j)\big)_{1\leqslant i,j\leqslant n} = V(x_1, \dots, x_n) \sum_{\lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0} b_\lambda P_\lambda(\underline{x}),$$

where the series converges for $|x_j| < r$ and

$$b_\lambda = \frac{1}{c_\lambda} \det\big(c^{(i)}_{\lambda_j+n-j}\big)_{1\leqslant i,j\leqslant n}.$$

Proof. We follow the proof of a similar formula for monomials x^m instead of $p_m(x)$, given in [6, p. 17]. We have

$$\begin{aligned}
\det\big(f_i(x_j)\big)_{1\leqslant i,j\leqslant n} &= \sum_{\sigma\in S_n} \varepsilon(\sigma) f_1(x_{\sigma(1)}) \dots f_n(x_{\sigma(n)}) \\
&= \sum_{\sigma\in S_n} \varepsilon(\sigma) \left(\sum_{k_1=0}^{\infty} c_{k_1}^{(1)} p_{k_1}(x_{\sigma(1)})\right) \dots \left(\sum_{k_n=0}^{\infty} c_{k_n}^{(n)} p_{k_n}(x_{\sigma(n)})\right) \\
&= \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1}^{(1)} \dots c_{k_n}^{(n)} \sum_{\sigma\in S_n} \varepsilon(\sigma) p_{k_1}(x_{\sigma(1)}) \dots p_{k_n}(x_{\sigma(n)}) \\
&= \sum_{k_1>\cdots>k_n\geqslant 0} \sum_{\tau\in S_n} \varepsilon(\tau) c^{(1)}_{k_{\tau(1)}} \dots c^{(n)}_{k_{\tau(n)}} \det\big(\big(p_{k_i}(x_j)\big)_{1\leqslant i,j\leqslant n}\big) \\
&= \sum_{k_1>\cdots>k_n\geqslant 0} \det\big(c_{k_j}^{(i)}\big) \det\big(\big(p_{k_i}(x_j)\big)_{1\leqslant i,j\leqslant n}\big).
\end{aligned}$$

Changing the sum indices $k_i = \lambda_i + n - i$, we get

$$\begin{aligned}
\det\big(f_i(x_j)\big)_{1\leqslant i,j\leqslant n} &= \sum_{\lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0} \det\big(\big(c^{(i)}_{\lambda_j+n-j}\big)_{1\leqslant i,j\leqslant n}\big) \det\big(\big(p_{\lambda_i+n-i}(x_j)\big)_{1\leqslant i,j\leqslant n}\big) \\
&= V(x_1, \dots, x_n) \sum_{\lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0} b_\lambda P_\lambda(x_1, \dots, x_n). \qquad \square
\end{aligned}$$

We give now an application of Proposition 3.6. Recall that the Legendre polynomials

$$P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m$$

form an orthogonal basis of $L^2(-1,1)$, i.e., μ is the Lebesgue measure restrained to $(-1,1)$. Denote by Q_m the Legendre functions of second kind. The Heine's formula [5, 3.10(10), p. 168] says that for $x \neq y$, $x, y \in \mathbb{R}$

$$\frac{1}{y-x} = \sum_{k=0}^{\infty}(2k+1)P_k(x)Q_k(y). \tag{13}$$

We will generalize this formula to the Hermitian matrix setting. For a partition λ, let P_λ and Q_λ be the corresponding symmetric functions defined by

$$P_\lambda(x_1,\dots,x_n) = \frac{\det(P_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)},$$

$$Q_\lambda(x_1,\dots,x_n) = \frac{\det(Q_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)}.$$

According to Corollary 3.2, the generalized Legendre polynomials $\hat{P}_\lambda$ form an orthogonal basis of the space $L^2_{U_n}(B(0,1), m(dx))$. The unit ball $B(0,1)$ is with respect to the norm $\|X\|$ of $X \in H_n$ considered as a linear functional on $\mathbb{R}^n$, i.e., $\|X\| = \max_i |x_i|$, where x_i are the eigenvalues of X.

Corollary 3.7. *Let $x_1,\dots,x_n, y_1,\dots,y_n \in \mathbb{R}$ be such that $x_i \neq y_j$, $i, j = 1,\dots,n$. Then*

$$\prod_{i,j=1}^{n}\frac{1}{y_i-x_j} = (-1)^{\frac{n(n-1)}{2}}\sum_{\lambda_1\geqslant\dots\geqslant\lambda_n\geqslant 0} d_\lambda P_\lambda(\underline{x})Q_\lambda(\underline{y}),$$

where

$$d_\lambda = \prod_{i=1}^{n}(2(\lambda_i+n-i)+1).$$

Proof. We apply Proposition 3.6 to the functions

$$f_i(x) = \frac{1}{y_i-x}$$

and we use the expansion (13), thus we have $c_k^{(i)} = (2k+1)Q_k(y_i)$. We get that

$$\det\left(\frac{1}{y_i-x_j}\right)_{i,j} = V(x_1,\dots,x_n)V(y_1,\dots,y_n)\sum_{\lambda_1\geqslant\dots\geqslant\lambda_n\geqslant 0} d_\lambda P_\lambda(\underline{x})Q_\lambda(\underline{y})$$

with $d_\lambda = \prod_{i=1}^{n}(2(\lambda_i+n-i)+1)$. By the Cauchy's determinant formula (see, e.g., [15, 7.6])

$$\det\left(\frac{1}{y_i - x_j}\right)_{i,j} = V(-\underline{x})V(\underline{y}) \prod_{i,j=1}^{n} \frac{1}{y_i - x_j}$$
$$= (-1)^{\frac{n(n-1)}{2}} V(\underline{x})V(\underline{y}) \prod_{i,j=1}^{n} \frac{1}{y_i - x_j},$$

and we get the expansion formula of the corollary. □

4. Examples

Example 4.1. Hermite polynomials.

Let us consider the family $\{h_m\}_{m\in\mathbb{N}}$ of monic Hermite orthogonal polynomials in $\mathbb{R}$. They are orthogonal polynomials with respect to the measure $\gamma(dy) = \mathrm{e}^{-y^2}\,dy$. According to (7) and Proposition 3.3, we define the monic Hermite polynomials on H_n by

$$\hat{H}_\lambda(X) = \frac{\det(h_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)},$$

where $x_1,\dots,x_n$ are the eigenvalues of the matrix X. They form an orthogonal basis of the space $L^2_{U_n}(H_n, \mathrm{e}^{-\mathrm{tr}(X^2)}\,dm(X))$, cf. Corollary 3.2. Recall that the condition "H_λ monic" means that

$$H_\lambda = \sum_{\kappa \preceq_{gl} \lambda} c_{\kappa\lambda} S_\lambda,$$

with $c_{\lambda\lambda} = 1$, cf. (9), and also that the leading term coefficient in the monomial decomposition (11) of H_λ is equal to 1.

In [12] Lassalle also considered generalized Hermite polynomials, but he used decompositions in the normalized Schur functions basis $S^*_\lambda = S_\lambda/S_\lambda(1^n)$, where $1^n = (1,\dots,1)$. Let us call H^*_λ the Hermite polynomials in [12]. It follows from [12, (i) p. 580] that the family $\{H^*_\lambda\}$ is obtained from the Schur polynomials $\{S_\lambda\}$ ordered in the graded lexicographic order, by the Gram–Schmidt orthogonalization process. The same is true for the family $\{H_\lambda\}$. Thus H^*_λ and H_λ differ only by a non-zero factor. If ones requires H^*_λ to be monic in the basis S^*_λ, we have $H^*_\lambda = c_\lambda H_\lambda$, with $c_\lambda = 1/S_{\lambda(1^n)}$. In this way we prove Theorem 6 of [12], communicated by the author without proof.

Also, if $h^{(a)}_m$ are monic Hermite polynomials generalized in the sense of Chihara, i.e. they are orthogonal with respect to the measure $\gamma_a(dy) = |y|^{2a}\mathrm{e}^{-y^2}\,dy$, $a \geqslant 0$, then the monic Hermite–Chihara polynomials of Hermitian matrix argument are defined by the formula

$$\hat{H}^{(a)}_\lambda(X) = \frac{\det(h^{(a)}_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)},$$

where $x_1,\dots,x_n$ are the eigenvalues of the matrix X. By Corollary 3.2, this family forms an orthogonal basis of the space $L^2_{U_n}(H_n, |\det X|^{2a}\mathrm{e}^{-\mathrm{tr}(X^2)}\,dm(X))$.

Example 4.2. Laguerre polynomials.

The Laguerre polynomials $L_m^{(\alpha)}$, $\alpha > -1$, are orthogonal polynomials on $(0,\infty)$ with respect to the measure $\mu_\alpha(dy) = y^\alpha \mathrm{e}^{-y}\mathbf{1}_{(0,\infty)}(y)\,dy$. Let us normalize them by the condition $L_m^{(\alpha)}(0) = 1$. Then they have the following explicit representation, see [18, (5.1.6) and (5.1.7)],

$$L_m^{(\alpha)}(y) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-y)^k}{(\alpha+1)_k}.$$

The Laguerre polynomials $\hat{L}_\lambda^{(\alpha)}(X)$ of Hermitian matrix argument are given, according to formula (7), by

$$\hat{L}_\lambda^{(\alpha)}(X) = c_\lambda \frac{\det(L_{\lambda_i+n-i}^{(\alpha)}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)} \tag{14}$$

where $x_1,\dots,x_n$ are the eigenvalues of the matrix X.

The measure $M_\alpha(dX) = (\det X)^\alpha \mathrm{e}^{-\operatorname{tr} X}\prod_i \mathbf{1}_{(0,\infty)}(x_i)\,dm(X)$, corresponding to μ_α via formula (3), is supported on the cone H_n^+ of non-negative definite Hermitian matrices. The polynomials $\hat{L}_\lambda^{(\alpha)}$ form an orthogonal basis of $L^2_{U_n}(H_n^+, M_\alpha)$. We normalize them by setting $L_\lambda(\underline{0}) = 1$. The constant c_λ may be determined using Proposition 3.4, for the coefficients $c_k^{(m)} = \binom{m}{k}\frac{(-1)^k}{(\alpha+1)_k}$. We compute

$$\begin{aligned}
\det\bigl(c_{n-j}^{(\lambda_i+n-i)}\bigr)_{1\leqslant i,j\leqslant n} &= \det\left(\binom{\lambda_i+n-i}{n-j}\frac{(-1)^{n-j}}{(\alpha+1)_{n-j}}\right)_{1\leqslant i,j\leqslant n}\\
&= \det\left(\frac{(\lambda_i+n-i)!}{(\lambda_i-i+j)!}\right)_{1\leqslant i,j\leqslant n}\prod_{j=0}^{n-1}\frac{(-1)^j}{(\alpha+1)_j\,j!}.
\end{aligned}$$

Set $t_i = \lambda_i + n - i$. Then

$$R_j(t_i) := \frac{(\lambda_i+n-i)!}{(\lambda_i-i+j)!} = \frac{t_i!}{(t_i-n+j)!} = t_i(t_i-1)\dots(t_i-n+j+1)$$

when $j < n$. For $j = n$ we have $R_n = 1$. Thus

$$\det\left(\frac{(\lambda_i+n-i)!}{(\lambda_i-i+j)!}\right)_{1\leqslant i,j\leqslant n} = \det\bigl(R_j(t_i)\bigr)_{1\leqslant i,j\leqslant n},$$

where the polynomials R_j have the degree $n-j$ and are monic. Multilinearity properties of determinant imply that $\det(R_j(t_i))_{1\leqslant i,j\leqslant n} = \det(t_i^{n-j})_{1\leqslant i,j\leqslant n}$, so it is equal to the Vandermonde determinant in t_i,

$$V(t_1,\dots,t_n) = \prod_{1\leqslant i<j\leqslant n}(t_i-t_j) = \prod_{1\leqslant i<j\leqslant n}(\lambda_i-\lambda_j-i+j).$$

Thus we find

$$c_\lambda = (-1)^{\frac{(n-1)n}{2}}\frac{\prod_{j=0}^{n-1}(\alpha+1)_j\,j!}{\prod_{1\leqslant i<j\leqslant n}(\lambda_i-\lambda_j-i+j)}.$$

As $\prod_{j=0}^{n-1}(\alpha+1)_j j! = \prod_{1\leqslant i<j\leqslant n}(\alpha+j-i)i$, we can also write

$$c_\lambda = (-1)^{\frac{(n-1)n}{2}} \prod_{1\leqslant i<j\leqslant n} \frac{(\alpha+j-i)i}{\lambda_i-\lambda_j-i+j}.$$

Formula (14) for generalized symmetric Laguerre polynomials, with c_λ given by the last equality, was announced in [14, Théorème 6], without proof.

Example 4.3. Jacobi polynomials.

For $a>-1$ and $b>-1$, the classical Jacobi polynomials $P_m^{(a,b)}$ are orthogonal polynomials on $[-1,1]$ with respect to the measure $\nu_{a,b}(dx) = (1-x)^a(1+x)^b\mathbf{1}_{[-1,1]}(x)\,dx$. We will consider a related family of Jacobi polynomials $J_m^{(a,b)}(y) := P_m^{(a,b)}(1-2y)$. The polynomials $J_m^{(a,b)}(y)$ are orthogonal on $[0,1]$, with respect to the measure $\mu_{a,b}(dx) = y^a(1-y)^b\mathbf{1}_{[0,1]}(y)\,dy$. If they are normalized by the condition $J_m^{(a,b)}(0)=1$, then, using [18, (4.21.2)], they have the monomial representation

$$J_m^{(a,b)}(y) = \sum_{k=0}^m \frac{(m+a+b+1)_k}{(a+1)_k}\binom{m}{k}(-y)^k.$$

The Jacobi polynomials $\hat{J}_\lambda^{(a,b)}(X)$ of Hermitian matrix argument are given, according to formula (7), by

$$\hat{J}_\lambda^{(\alpha)}(X) = c_\lambda \frac{\det(J_{\lambda_i+n-i}^{(a,b)}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)} \tag{15}$$

where $x_1,\dots,x_n$ are the eigenvalues of the matrix X.

The measure $M_{(a,b)}(dX) = (\det X)^a(\det(I-X))^b\prod_i \mathbf{1}_{[0,1]}(x_i)\,dm(X)$ corresponding to $\mu_{a,b}$ via formula (3), is concentrated on the intersection $H_n^+ \cap \bar{B}(0,1)$ of the cone H_n^+ with the unit ball $\bar{B}(0,1)$ in H_n. The polynomials $\hat{J}_\lambda^{(a,b)}$ form an orthogonal basis of $L^2_{U_n}(H_n^+\cap\bar{B}(0,1), M_{(a,b)})$. We normalize them by setting $J_\lambda^{(a,b)}(\underline{0})=1$. We apply Proposition 3.4 in order to compute the constant c_λ in (15). The coefficients $c_k^{(m)}$ are equal $c_k^{(m)} = \frac{(m+a+b+1)_k}{(a+1)_k}\binom{m}{k}(-1)^k$. We obtain

$$\begin{aligned}
&\det\bigl(c_{n-j}^{(\lambda_i+n-i)}\bigr)_{1\leqslant i,j\leqslant n}\\
&\quad= \det\left(\frac{(\lambda_i+n-i+a+b+1)_{n-j}}{(a+1)_{n-j}}\binom{\lambda_i+n-i}{n-j}(-1)^{n-j}\right)_{1\leqslant i,j\leqslant n}\\
&\quad= \det\left(\frac{(\lambda_i+n-i+a+b+1)_{n-j}(\lambda_i+n-i)!}{(\lambda_i-i+j)!}\right)_{1\leqslant i,j\leqslant n}\\
&\qquad\times\prod_{j=1}^n \frac{(-1)^{n-j}}{(a+1)_{n-j}(n-j)!}.
\end{aligned}$$

Setting $t_i = \lambda_i + n - i$ and $A = a + b + 1$, the previous determinant can be written as

$$D := \det\left(\frac{(t_i + A)_{n-j} t_i!}{(t_i - n + j)!}\right)_{1 \leqslant i, j \leqslant n} = \det\left(\prod_{m=0}^{n-j-1} (t_i + A + m)(t_i - m)\right)_{1 \leqslant i, j \leqslant n}.$$

Taking $t_i = t_j, 1 \leqslant i, j \leqslant n$, this determinant vanishes and therefore it is divisible by $\prod_{1 \leqslant i < j \leqslant n}(t_i - t_j)$. If we take $t_i = -t_j - A, 1 \leqslant i, j \leqslant n$, the determinant also vanishes, so it is divisible by $\prod_{1 \leqslant i < j \leqslant n}(t_i + t_j + A)$. Thus

$$D = \left[\prod_{1 \leqslant i < j \leqslant n} (t_i - t_j)(t_i + t_j + A)\right] R(t_1, \ldots, t_n),$$

where R is a polynomial. When we fix $t_2, \ldots, t_n$ and consider D as a polynomial of t_1, we see that it is monic and of degree $2n - 2$. The same is true for $\prod_{1 \leqslant i < j \leqslant n}(t_i - t_j)(t_i + t_j + A)$. Thus the polynomial R does not depend on t_1. Repeating this argument for all t_i we deduce that $R = 1$. Finally we get

$$c_\lambda = (-1)^{\frac{(n-1)n}{2}} \prod_{1 \leqslant i < j \leqslant n} \frac{(\alpha + j - i)i}{(\lambda_i - \lambda_j - i + j)(\lambda_i + \lambda_j + 2n - i - j + a + b + 1)}.$$

Formula (15) for generalized symmetric Jacobi polynomials, with c_λ as in the last equality, was given without proof in [13, Théorème 10].

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